A Prelude to Calculus: History and Pedagogy

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Introduction

Problem

The concept of a limit which is fundamental to calculus is a radically new concept for students who are studying it for the first time. The problem to be explored here is how to introduce this new concept to students in such a way that they have time to ponder and eventually assimilate it. The researcher's opinion of the standard method is that while it makes reference to graphs of curves and how in the limit their secants become tangents, the method fails to demonstrate to students how this can be used and why it is important, and proceeds almost immediately to a series of algebraic expressions of which the limit is to be found.

Proposed Solution

The solution proposed here is to take a historical perspective to the development of calculus with a focus on Archimedes' Method of Equilibrium for the volume of a sphere so that students are able to understand how people could have used the idea of limits to solve a real problem. The idea of a limit is essential in Archimedes' method and while carefully working through this technique, students' experiences will be firmly grounded in their own understanding as they recreate the historical development of calculus.

Focus

The primary focus of this paper will be on students' conceptual development of limits in response to the intervention described below. A secondary focus will be on the change in students' thinking about the nature of mathematics as well as their affect toward the subject of math in general.

Background and Literature Review

The invention of calculus was one of the biggest turning points in the history of mathematics. For thousands of years many mathematicians had explored the ideas of the infinite and infinitesimals, but it was Newton and Leibniz who are credited with finally generalizing the concept of a limit which led to the formalization of differentials and integrals. Most, if not all students who arrive on the doorstep of calculus, are generally unaware of the path that these ideas have trod to arrive in their textbook. The researcher sampled three calculus texts (Willcox, Buck, Jacob, & Bailey, 1971; Seeley, 1973; Edwards & Penney, 2002) and the typical approach is to present the concept of a limit as an established routine disconnected from practical application. Though one source was found which does introduce limits within the context of a practical problem (Greenspan & Beeney, 1973, p.2-4), besides a mere mention of Archimedes, no historical context is present. The scheme to be explored here is taken from Postman (1979, p.138) who suggests that all subjects be taught as history such that students may gain a fuller appreciation for them, and in so doing come to grips with how each subject came to be what it is today.

Most calculus courses start with a treatise on limits, as they should, since the concept of a limit is fundamental in understanding both differentials and integrals. The lesson described below is meant to be situated at the very beginning of a calculus course to outline the milieu out of which the concept of the limit was invented, since, without such a background, 'for the student…the introduction of the limit concept suddenly appears for no reason, with all the cognitive problems this may bring.' (Gravemeijer & Doorman, 1999, p.113). What's worse is that students are not normally given the opportunity to explore the finer philosophical grounds on which limits are based and which confounded mathematicians for millennia. The intervention described below will address that issue, but it will also provide an answer to the age old student question of 'when will I ever use this?' by at least showing how the ancients used these ideas to solve practical problems.

An emphasis during this lesson will be on the discussion of the pertinent ideas amongst the students and the teacher since according to Paul Ernest 'in deep and multiple ways…mathematics is at base conversational.' (1994, p.35). This is essential to gaining a fuller understanding of mathematics since as Ernest goes on to say 'discourse and language…play an essential role in the genesis, acquisition, communication, formulation and justification of virtually all knowledge, including, in particular, mathematical knowledge.' (1994, p.37).

It will be argued by some that we need not waste time presenting and discussing obscure cases in the history of mathematics and that we should proceed immediately to the important results (Mathematically Correct, n.d.), but this line of argument underscores an even more important issue to do with the nature of mathematics; namely, what is it? Is it something that the absolutists would describe as an inherent feature of nature; eternal, unchanging, waiting to be discovered, or would it be better described by the fallibilists, as something pliable, subject to change, and an invention of the human mind? Hellman (2006, p.214) references a study done at Queen's University in Kingston, Canada that found that most students' image of mathematics is a result of the teaching method used in their class which tends to the absolutist point of view. This belief is widely held by many teachers in Australia, since, as Archer (2000, p.9) says: 'most of the secondary mathematics teachers [seem] to view mathematics as more self-contained, a set of logical relationships that [exist] in abstract form almost divorced from the everyday lives of students.'

Gravemeijer and Doorman illustrate a typical approach to presenting a mathematical topic as breaking it up into pieces whose chronology, in the eyes of the mathematician, lead to a global understanding of that topic (as cited in Tall, 1999). However they go on to warn that this methodology is extremely hazardous since the students may not make the connections between all the pieces, but furthermore, 'it may be even worse, if the student does not realize that there *is* a big picture.' (Gravemeijer & Doorman, 1999, p.112). Despite this potential limitation, if students can begin to see mathematics as ultimately a study of the human mind, then they will have a good chance of relating ideas that they once saw as independent, but most importantly, if they know some of the history of mathematics, they can understand the evolutionary processes for the topics they study, and in so doing, begin to appreciate that the laconic theorems that constitute nothing more than lines in their texts, were produced by hundreds of years of discussion by many of the world's brightest minds. This in itself may help students overcome the gravest misconception of all which is the feeling of unworthiness when they don't immediately grasp a difficult topic which their textbook presents so matter-of-factly. The extent of this lack of self-belief has been well documented (e.g. Barrington, 2006; Thomson & Fleming, 2002).

Postman's (1979) view that all subjects should be taught as history was driven by the belief that a fundamental purpose of education should be to provide whatever society is lacking. He also believed that the most salient of those things lacking was continuity in people's worldview due to the effects of mass media's continual emphasis on the present. The researcher's own belief that students enjoy learning within a subject's proper historical context came from a normally rowdy grade eleven mathematics class when the topic of arc length and sector area was presented as the story of Eratosthenes' calculation of the circumference of the earth (Diggins, 1965). Surprisingly, everyone was attentive and ostensibly impressed by this truly incredible story. Gravemeijer and Doorman describe this as a 'perspective [which] is consistent with a more general view that the way in which mankind developed mathematical knowledge is also the way in which individuals should acquire mathematical knowledge.' (1999, p.116).

Method

Overview

This is a case study of a participatory action research (Kemmis & McTaggard, 2000) project with a single student.

Participant

The student participant in this lesson was a year eleven girl named Lisa (pseudonym) who attends a maths and sciences specialist school in Queensland. She is taking the International Baccalaureate program, is quite bright, highly motivated, and receives decent marks in math. The researcher has spent many lessons tutoring Lisa in mathematics and though she needs help with some higher order problem solving questions, she can usually do those which require only knowledge and procedures on her own. In addition to the heavy emphasis on math and science in her school program, she also takes a class called Theory of Knowledge which involves critical discussion on the nature of the material studied in school which includes mathematics.

Intervention

The intervention proceeded as described below.

Zeno's Paradoxes (ca. 450 B.C.)

The intervention began by posing the question to the student: "Is a given quantity infinitely divisible, or is it made up of a finite number of indivisible parts?" This question paves the way for Zeno's famous paradox of the dichotomy in the fifth century B.C. The paradox goes that if a line segment is infinitely divisible, then if one wants to traverse the distance of that line segment, one first needs to travel one-half the distance, but in order to get there, it is necessary to travel one-quarter of the way, and to get there, first to reach one-eighth of the way and so on. The result is that motion can never even begin since one can keep dividing the distance to be traveled in half *ad infinitum*. Of course, the paradox arises since motion *is* possible. (Eves, 1990, p.379-380). This idea of dividing a quantity as many times as we like is the essence behind the idea of a limit. The paradox was posed to the student, citing Zeno and the historical period, and a small discussion ensued. The point emphasized was that this idea was one that the Greeks argued over for centuries and people continue to do so today – there is no simple answer.

Archimedes' Method of Equilibrium (ca. 287-212 B.C.)

The lesson then proceeded to sketch Archimedes' Method of Equilibrium which he used to calculate the volume of a sphere. This process is summarized nicely by Eves (1990, p.383-385). Also known as the Law of the Lever, it says that given masses M and m, and distances D and d, then the lever will be balanced when MD = md (see Figure 1). This product of mass and distance is called the 'moment' in physics.

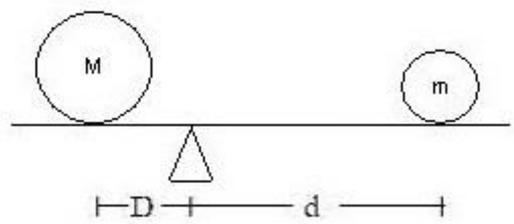

Figure1. Archimedes' Law of the Lever

This is a familiar concept to any physics student, but it was worth taking the time to use a scale and some weights to test its validity with the student since she had not experienced the phenomenon first hand. This idea led to Archimedes famous aphorism 'Give me a place to stand on, and I can move the earth.' (Heath, 1953, p. xix). Archimedes' idea was that, assuming uniform density, we can calculate the volume of a solid if we know the volume of another one, and can balance them on a lever as shown above. He then proposes creating a sphere, cylinder and cone of the relative dimensions shown below in Figure 2.

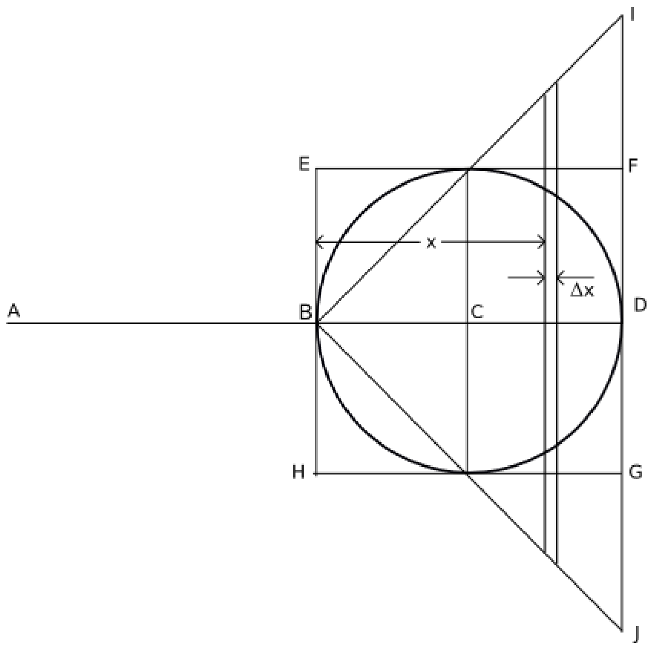

Figure 2. Cross-sections of a sphere centered at C, the cylinder EFGH, also centered at C, and the cone BIJ. It should also be noted that the distance AB = BD.

To sketch out where we are going with this diagram, we are going to treat the line AD as a lever, with point B being the fulcrum. Archimedes' reasoning then goes that if we were to take a very thin section of each of the three solids, denoted by Δx, then we can approximate these cross-sections as very thin cylinders, whose volumes we can calculate. At this point, the lesson paused and Lisa was asked if she thought this was a fair approximation. The literature has little to say about how students view this problem. Archimedes argued that if we just make Δx small enough then the difference between the approximation and the true value will become negligible. Recall Zeno's paradox of the dichotomy. This argument is in line with that view that a volume can be divided an infinite number of times. Figure 3 gave the student a close up of how the argument for making Δx smaller and smaller better approximates the sphere.

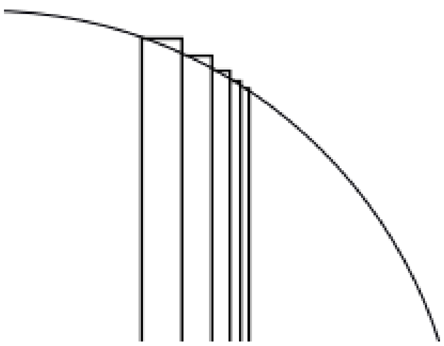

Figure 3.

If some students are unconvinced by this argument, it is suggested to ask them to suspend their disbelief and assume for a moment that we can make such an approximation in order to see what the results of such an assumption are. This was exactly the course of action taken with Lisa. Noting that the sphere's radius is equal to BC, which is also equal to DC, and denoting this ‘r’, the volumes of each small cross-section are then:

$$V_{cylinder} = \pi \cdot r^2 \cdot \Delta x \quad V_{cone} = \pi \cdot x^2 \cdot \Delta x \quad V_{sphere} = \pi \cdot x \cdot (2r - x) \cdot \Delta x$$

Pythagorean theorem is used to come up with the expression for the radius of the cross-section of the sphere. Archimedes then proceeds to hang the slices of both the sphere and the cone at point A which is 2r from the fulcrum, which recall, he took to be at point B. After working out the product of these volumes and their distance 2r from point B, this gives a total moment of:

$$4\pi \cdot x \cdot r^2 \cdot \Delta x$$

The section of the cylinder is left hanging at a distance ‘x’ from the fulcrum. The moment of it is:

$$\pi \cdot x \cdot r^2 \cdot \Delta x$$

This setup is illustrated in Figure 4 below.

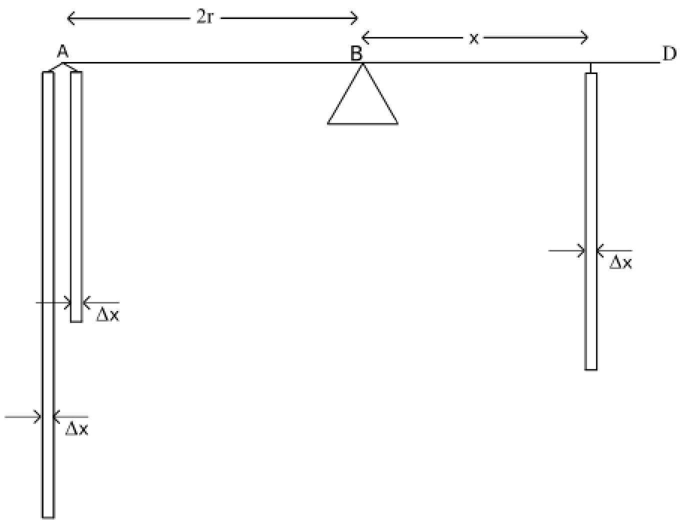

Figure 4. The cross-sections of the sphere and cone hanging at A, and the cross-section of the cylinder hanging at a distance ‘x’ from point B.

Archimedes then notes that the combined moment of the sections of the sphere and cone is four times the moment of the slice of the cylinder when left where it is. At this point the intervention was paused and a reminder was given to Lisa about what we were doing, why, and the assumptions we had made since the next step is a crucial one.

Archimedes then argues that if we add up *all* such slices for each solid we can approximate the volume for the entire solid. The distance x where the cross-sections were chosen was arbitrary, and thus the above relations hold for all values of x. At this point, students should be asked if they feel this is a good approximation and why or why not and this is exactly the question which was posed to Lisa. Quite possibly, many students will argue that it is not a good approximation and we should encourage such critical thinking. Again though, we can ask students to just wait and see what happens if we do make such an approximation. The result of this reasoning is that if we add up all such cross-sections, thus forming the entire sphere, cone, and cylinder, and hang the entire sphere and cone at point A, their moment is four times that of the moment of the entire cylinder when left where it is:

2r[volume of sphere + volume of cone] = 4r[volume of cylinder]

From here the rest of the proof is an algebraic one, but first there are two small diversions that were taken in order to ensure that Lisa understood the reasoning. The first has to do with the idea of centre of gravity such that we can treat the entire cylinder as hanging from its centre, point C. One way this can be done is by referring back to the scales again, and asking the students how we can treat the weights as acting from one specific distance. Using the scales, Lisa, under guidance by the researcher, demonstrated that because the weight is symmetrical about the vertical, each point will counter balance the point on the opposite side of the vertical. She could then see that we can treat the entire mass of the object as acting through the vertical axis of symmetry – or, through the centre of mass.

The next point that we need to address is, how did Archimedes know the volume of a cone? After all, in order to use the formula we found above, it is essential to know the volume of the cone to find the volume of the sphere. The question was posed to the student to let her formulate her own ideas. The volume of a cone seems to have been found by Democritus (O'Connor & Roberstson, 1999, ¶27), who determined it to be $\frac{1}{3}\cdot\pi\cdot h\cdot r^2$, where h is the vertical height and r the radius of the base, which he is suspected of finding by a method similar to the one Archimedes is using, but we will have to leave it to the students' imagination as to how exactly Democritus found it.

Substituting into the above equation:

2r[volume of sphere + volume of cone] = 4r[volume of cylinder]

we find:

$$[\text{volume of sphere} + \frac{1}{3}\cdot(2r)\cdot\pi\cdot(2r)^2\,] = 2[\,(\pi\cdot r^2\cdot(2r)\,]$$

and solving for the volume of a sphere we find:

$$\text{volume of a sphere} = \frac{4}{3}\cdot\pi\cdot r^3$$

which is of course the formula that students are familiar with. Lisa was given a few minutes to reflect on the result and line of argument used. The final step was telling her that what had just been accomplished was actually *not* Archimedes proof of the volume of a sphere, but rather merely a method used to come up with an approximation of the equation. The actual rigorous proof was done by the Greek Method of Exhaustion and could be a whole other lesson in itself.

Data Sources

The student was interviewed prior to the intervention about her ideas of the nature and development of mathematics as well as her affect toward the subject. A copy of the questions posed and answers Lisa gave can be found in Appendix A. A digital sound recorder was used to record all conversation throughout the intervention and all paper which was used for calculations was kept. A transcript of the researcher and student discourse was made and the student was interviewed after the intervention to determine any changes in thinking that had occurred. A copy of the questions posed and answers Lisa gave after the intervention can be found in Appendix B.

Data and Analysis

Intervention

1. In response to Zeno's paradoxes, the following discussion took place:

Researcher: "do you think it's possible to divide a quantity up an infinite number of times or do you think that the quantity is composed of a finite number of indivisible parts"
Lisa: "umm, infinite on a scale, like, when you get down to atoms and quarks and stuff like that. Going further than that I think it has its limits."
Researcher: "OK so you think there's a limit to how much you can divide a quantity up, we cannot divide infinitely – is that what you're saying?"
Lisa: "Oh it depends. If you're actually doing it then there's limits, but if you're doing it with numbers then you can use as many numbers as you want."

In this discussion we can see Lisa's understanding of two different processes – one of them mental and one of them physical. The nature of mathematics as something abstract is shown clearly when she admits that while there are limits to splitting a quantity in the real world, in a mathematical sense we can do it indefinitely. This seems to suggest that she believes that the human mind itself is something infinite but it's not clear in this dialogue whether or not such a belief leans toward an absolutist or fallibilist point of view.

2. Later when posed with the paradox of the dichotomy, Lisa responded:

Lisa: "It's confusing."
Researcher: "Why?"
Lisa: "Because you keep sort of thinking of it in the real world, because in the real world yes you are moving, literally."
Researcher: "Right."
Lisa: "But theoretically in your head, it's just a little hard to grasp the concept."
Researcher: "OK, so these ideas are ideas, and when we try and transfer them to the real world maybe there is some disparity between them?"
Lisa: "Mm, yeah. There are some limits, taking theoretical things into the real world."
…
Lisa: "We know what goes on in a black hole but we don't actually know it."
Researcher: "Right, so we have a theory about it but no one has ever been there and back."
Lisa: "Yeah."

This dialogue expands on that in number 1 and sheds some light on why Lisa and possibly some other students, may find the concept of limits so difficult. The idea of a limit is an abstract one, and only exists as such. We cannot transfer this process to the real world, which she said she keeps trying to do.

3. During the activity using the balance and weights, it took Lisa a couple of minutes to relate the concept of the balance to that of a symbolic equation. Following the activity which demonstrated the principle of equilibrium, the following exchange occurred:

Lisa: "Where does this come into the real world? Like how can you apply it?"
Researcher: "Why do you think they put door handles on the far end of the door?"
Lisa: "Oh, because it's easier to open it. It takes less energy."

Here Lisa shows yet again her desire to link concepts back to the real world, and once a familiar example is given, she assimilates the idea.

4. While working through the dissection of the solids into very thin cylindrical sections, Lisa managed to work through the cross-section of the solid cylinder, but when it came to finding the cross-section of the cone, the following conversation happened:

Researcher: "We're going to take a very very thin slice, what's the volume of that little section going to be?"
Lisa: "It'd be hard because that distance [pointing to the edge of the cone] isn't the same."
Researcher: "What do you mean it's not the same?"
[Lisa proceeds to draw a picture then refers to the edge as shown in Figure 5.]

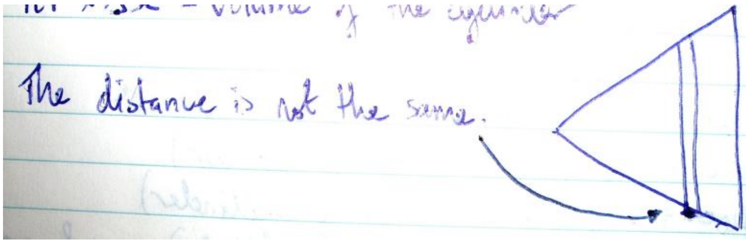

Figure 5.

Lisa: "It would sort of extend out and then down like that." [as shown where the arrow is pointing in Figure 5]
Researcher: "But how about if we could make it [the small part extending out from the cone as indicated by the arrow in Figure 5] just a little bit smaller, a little bit smaller, a little bit smaller."
Lisa: "Oh I get it."
…
Researcher: "Now what do you think about this line of argument? If we just make it small enough?"
Lisa: "It would work but it wouldn't…it still wouldn't give you the exact answer."
Researcher: "Why not?"
Lisa: "It's only just an approximation. A really good one but…"
…
Researcher: "So would you say it's a good approximation?"
Lisa: "It's a good approximation, but in the real world if it's absolutely necessary to get the exact amount then it's not very good to approximate."

It is now clear that Lisa has trouble conceptualizing the limit as something abstract due to her tendency to link everything back to the real world. Because of this, the concept of a limit is troublesome since the process of taking a limit as something becomes infinitesimally small cannot be carried over into the real world – it is an abstraction. Perhaps the weight and scale materials primed Lisa for something which could be understood completely with real examples, and in this way, it's possible that in future more emphasis needs to be placed on the fact that finding a limit is an abstract process.

It is possible that since Lisa views math as something fundamentally abstract and therefore as a function of the mind rather than a function of the real-world, her beliefs in math would better be described as fallibilist than absolutist, since ideas about the world are subject to error whereas the real-world itself is not.

5. Finally, at the end of the lesson when we had come up with the equation for the volume of a sphere by following the steps in the intervention, the following occurred:

Researcher: "What do you think about our result and our method?"
Lisa: "It's…complicated."
…
Researcher: "The method is certainly complicated, what do you think about the result?"
Lisa: "It's simple. Like, I don't know, the actual result is easy to use. It's just got one variable."
Researcher: "The method was complicated was but is it sound?"
Lisa: "Yeah because we've proved it…that it works. But, then again you've got all those little extra bits down the bottom of the side."

This exchange shows the great divide between Lisa's intuitions about the validity of the result based upon her prior use of the formula, "we've proved it", and her uncertainty with the method regarding "all those little extra bits". The reason for this is likely that she was first exposed to the formula for the volume of a sphere without a proof; it was just something presented to be memorized. However, when confronted with the Method of Equilibrium which provided a reasoned development of the formula, "all the little extra bits", referring again to that which the arrow is point in Figure 5, made her doubt the result.

6. During a brief discussion after the lesson the student responded to the researcher's questions as follows:

Researcher: "The relative length of time it took for things to appear and develop, what do you think about that?"
Lisa: "It's pretty cool considering they [the ancients] didn't have technology, they had their technology, and it's just like comparing really old stuff to new stuff and then saying that the old stuff seems to make more sense sometimes."
…
Researcher: "Like you said this method is kind of complicated and long. How long do you think it took people to improve upon this method?"
Lisa: "Um…it would take, a bit of time…for technology to develop and for the actual method to spread around to other people."

Researcher: "Do you want to hazard a guess?"
Lisa: "Um, more than a hundred years."
[It actually took about 1900 years before calculus was formalized by Newton and Leibniz]
…
Researcher: "anything else you want to say about anything here, anything we've done, any impressions, any thoughts, any feelings you want to put out there?"
Lisa: "Well, I'm pretty amazed about how they did it, and how I can understand it. It's cool."
Researcher: "It is kind of cool isn't it?"
Lisa: "And like, without numbers. I find stuff without numbers pretty hard."

This last exchange confirms Lisa's difficulty with the abstract part of math, since she "find[s] stuff without numbers pretty hard." It also reveals an appreciation, albeit an undervalued one, for the time needed for mathematics to develop. This point should underscore the fact that the theorems and formulas that appear in students' textbooks, are not as matter-of-fact as they appear, since, in this case, like many others, it took an enormous amount of time for the Method of Equilibrium to be improved upon. Though the researcher mentioned this large passage of time between the invention of Archimedes' Method and Newton and Leibniz's development of calculus, in future it could greatly benefit students by emphasizing the point about theorems and formulas not being so obvious to quell any feelings of unworthiness students may have when they don't immediately grasp a difficult concept.

Student's Response to Interview Questions

Part of the response Lisa gave to question 6 after the intervention was:

"I also liked working with the scales as they helped me to understand the Method of Equilibrium. I did not like having to picture the cross-sections of the shapes as I found it hard to picture them, especially the cylinder."

We can see that the concept of the limit as it relates to this lesson by taking very thin sections of the solids was difficult to imagine. This fits in with the discussion above regarding Lisa's difficulties with abstract concepts. Figure 3 was only partially helpful in illustrating this idea and in future other three dimensional models to help explain it may well be beneficial in helping students to visualize the method.

Lisa's answer to question 4 after the intervention was:

"I felt more confused than frustrated [about the Method of Equilibrium], as this was a completely new concept for me. I did not really know what to ask, as I was not familiar with the language used. I also found it difficult to picture the cross-sections of the shapes."

From this we can see that care needs to be taken with the language we use, particularly in describing new concepts. Also, when she says she felt confused, it is not clear whether she had enough time to really understand the entire lesson. In the intervention dialogue number 6 above, Lisa talks about how she is amazed she can understand the lesson but perhaps when she reflected later upon it and wrote her answers to the questions in Appendix B, she still had lingering doubts and uncertainties.

Lisa's concept of the nature of mathematics is quite fallibilist and was unchanged by this intervention as can be seen by the way she answered question 1 both before and after the lesson:

"I believe mathematics is a process that people use to understand the world around them using numbers, logic, and calculations. I think that it is just a "device" that has been invented and developed."

"I think that the Method of Equilibrium was invented and developed as the concepts were thought of. Archimedes had invented this method to explain a naturally occurring part of nature, a sphere."

Given her studies in her Theory of Knowledge course where there is a lot of critical discussion on the nature of mathematics, this sort of response is not too surprising. However, given that most students do not take such a course in school and also that many studies imply that absolutism is still the dominant view from which mathematics is taught (Hellman, 2006, p. 211), this lesson, based upon the historical methods described, may well encourage many students to reflect upon the absolutist nature of mathematics, particularly once they begin their formal study of limits in calculus and are able to compare it to Archimedes' Method of Equilibrium.

Lisa's response to question 4 prior to the intervention was:

"When I can't understand a concept in math I feel confused and frustrated. I think that the math that we are learning is pointless and I feel that I cannot ask for help as I don't know what to ask for because I just don't understand what I am supposed to do. I also believe that the way in which the math concepts are asked confuse me. I know that I can do the knowledge and procedure questions but when it comes to the modelling and problem solving questions that use multiple processes and reasoning I find that the processes I use get muddled up and I end up writing things that don't make sense logically or mathematically."

This demonstrates the deep frustration that students feel when unable to grasp mathematical concepts. Furthermore, in this case when she calls math "pointless" we can see the deep chasm that exists between meaningful learning and learning that exists entirely to gain good grades. As can be seen in her response to question 6 above, more meaningful learning took place when materials were involved (with the balance and weights), and the lesson could have benefited with other materials to teach the part about taking very thin sections, which is the essence of limits.

Lisa's response to question 5 after the intervention was:

"I think that other people would have felt confused and frustrated until they understood the concept [of the Method of Equilibrium]. They probably would have asked questions and kept asking questions until they fully understood the concept of the Method of Equilibrium."

This seems to suggest that more time would have been beneficial in exploring this radically new concept. It also suggests that this lesson is worthwhile inasmuch as students have at least some time to explore the conceptual side of limits whereas in the traditional approach the students are expected to automatically accept this new idea.

Student Reflections on the Learning Process

Lisa's overview of this lesson seems to indicate that the time spent was worthwhile in helping her to develop an understanding of part of the history of mathematics and the new idea of using limits to solve problems. Below is part of her response to question 6 after the lesson:

"I liked the historical aspect of the lesson and talking about how interesting it is that Archimedes thought of this method without the use of today's technology….On the whole I enjoyed learning about the Method of Equilibrium."

This suggests that in future other students could benefit from this lesson and from this discussion- and exploratory-based pedagogy from a historical perspective.

Conclusions

The problem posed in this paper was how to introduce the new concept of a limit to students in such a way that they have time to ponder and eventually assimilate it. Central to the solution was a historical approach to the development of calculus with a focus on Archimedes' Method of Equilibrium so that the idea of limits would be seen to solve a real problem within a real historical context. While working through this approach the student connected her own experiences to those actually played out in the historical development of calculus.

The lesson bore out that the new concept of a limit was difficult for this student to understand, but her affinity for the historical approach confirmed the findings of Gravemeijer and Doorman that 'context problems are rooted in this reality, on the other hand, solving these context problems helps the students to expand their reality.' (1999, p.127). Lisa's confusion about the concept of a limit suggests that more time spent on the topic would be beneficial. However, of equal and perhaps more importance is presenting the lesson in a whole-class environment since this would generate more discussion which is significant since 'in deep and multiple ways…mathematics is at base conversational.' (Ernest, 1994, p. 35). Of equal importance however, is the need in future to use materials which depict the sections of the solids more clearly in order to help the students picture the process of dividing them up. One possibility is to use some play dough to model the solids and then actually cut them up. Students with different learning styles including those who are challenged in visualizing the conversion of three dimensional models to two dimensional representations may well profit greatly from this activity.

The student's view of the nature of mathematics was an unchanged fallibilist one, but in light of her Theory of Knowledge class which deals with the nature of math, it is quite likely that the sort of approach described here would benefit many other students since such a class is not offered in typical schools and because 'the battle between the absolutists and fallibilists goes on…perhaps most furiously in the world of mathematics education….[and] a variety of studies seem to show that absolutism is still the majority point of view.' (Hellman, 2006, p. 211).

The student's affect toward mathematics confirmed what has been shown by many studies (e.g. Barrington, 2006; Thomson & Fleming, 2002); namely, a feeling of frustration toward and confusion about difficult topics. It is noted however, that Lisa found this approach interesting and it possible that given more time and discussion with peers, the confusion which she referred to would be lessened.

Though it will likely be clear how the ideas presented in this lesson relate to the formal concept of the limit in subsequent classes, further development could make the formal treatment of limits refer back to this lesson in order to make the links explicit, ideally with more historical references. Overall, it seems that due to Lisa's positive reaction to learning mathematics from a historical perspective, other students might react the same way. Further research on presenting this lesson in a whole-class environment would reveal whether or not this is true.

Appendix A: Questions posed before the lesson, with student responses.

Q1: What do you believe mathematics is? Describe it any way, with words or pictures, that you feel best expresses your beliefs.

A1: I believe mathematics is a process that people use to understand the world around them using numbers, logic, and calculations. I think that it is just a "device" that has been invented and developed. There are some fields in math that have been invented and developed for example algebra, but some of the processes within it have been discovered such as natural fractals and numbers such as pi and Euler's number.

Q2: What do you think, if any, the limits of mathematics are?

A2: I believe that limitations depend on the context and language that mathematics is communicated in and the interpretation of the question. For example how many sides does a circle have? Someone's answer could be 2, an inside and an outside whereas someone else's interpretation might lead them to an answer of an infinite amount of sides. The problem with this question would lie with the people's interpretation of what a "side" is. I think that the limitations are not of mathematics itself, but the way that it is used and communicated.

Q3: How do you think mathematics developed?

A3: I think that how mathematics is used has developed because of the development of technology, like for example from slide rulers to calculators to eventually the quantum computer. As for actual mathematics, I believe that it has developed and branched out into new fields such as fractal geometry and I believe that technology has also played a part with this developmental process.

Q4: How do you feel when you cannot grasp a difficult concept in mathematics? Describe your thoughts or feelings in any way, with words or pictures, that you feel best expresses them.

A4: When I can't understand a concept in math I feel confused and frustrated. I think that the math that we are learning is pointless and I feel that I cannot ask for help as I don't know what to ask for because I just don't understand what I am supposed to do. I also believe that the way in which the math concepts are asked confuse me. I know that I can do the knowledge and procedure questions but when it comes to the modelling and problem solving questions that use multiple processes and reasoning I find that the processes I use get muddled up and I end up writing things that don't make sense logically or mathematically.

Appendix B: Questions posed after the lesson, with student responses.

Q1: In the prior questions, you said that mathematics has been invented and developed, but in some aspects of mathematics, certain processes have been discovered. What do you believe the nature of Archimedes' Method of Equilibrium is?

A1: I think that the Method of Equilibrium was invented and developed as the concepts were thought of. Archimedes had invented this method to explain a naturally occurring part of nature, a sphere.

Q2: What do you think, if any, the limits of Archimedes' Method of Equilibrium is?

A2: I think that the person using this method creates the limits of the Method of Equilibrium. I found it difficult to think of the concept of infinite slices as I cannot imagine or picture it in my head.

Q3: With regards to Archimedes' Method of Equilibrium, How do you think mathematics developed?

A3: I think that math has developed to include different concepts and different fields. In regards to the Method of Equilibrium, I believe that the thought processes involved have developed but the method of undertaking the problem has not.

Q4: How did you feel when you couldn't grasp some of the concepts or procedures in Archimedes' Method of Equilibrium? Describe your thoughts or feelings in any way with words or pictures, that you feel best expresses them.

A4: I felt more confused than frustrated, as this was a completely new concept for me. I did not really know what to ask, as I was not familiar with the language used. I also found it difficult to picture the cross-sections of the shapes.

Q5: With respect to Question #4, how do you think others may feel when encountering Archimedes' Method of Equilibrium, or may have felt historically when encountering this Method?

A5: I think that other people would have felt confused and frustrated until they understood the concept. They probably would have asked questions and kept asking questions until they fully understood the concept of the Method of Equilibrium.

Q6: What did you like or dislike about this lesson, and why? Finally, give any other thoughts or feelings with regards to this lesson that may not have been covered in the above questions.

A6: I liked the historical aspect of the lesson and talking about how interesting it is that Archimedes thought of this method without the use of today's technology. I also liked working with the scales as they helped me to understand the Method of Equilibrium. I did not like having to picture the cross-sections of the shapes as I found it hard to picture them, especially the cylinder. The algebra was all right as it was not too complicated. On the whole I enjoyed learning about the Method of Equilibrium.

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